

Duality in Quantum Information Geometry

R. F. Streater, Dept. of Mathematics, King's College London

9 June 2003

Abstract

Let \mathcal{H} be a separable Hilbert space. We consider the manifold \mathcal{M} consisting of density operators ρ on \mathcal{H} such that ρ^p is of trace class for some $p \in (0, 1)$. We say $\sigma \in \mathcal{M}$ is nearby ρ if there exists $C > 1$ such that $C^{-1}\rho \leq \sigma \leq C\rho$. We show that the space of nearby points to ρ can be furnished with the two flat connections known as the (\pm) -affine structures, which are dual relative to the *BKM* metric. We furnish \mathcal{M} with a norm making it into a Banach manifold.

Keywords: Infinite dimensional quantum information manifolds with dually flat affine connections.

1 Introduction

1.1 The problem

The set of states of a classical or quantum system can be furnished with the L^1 -norm relative to a reference measure (or the trace norm for quantum systems). The associated metric is not the ideal measure of distance between states. For example, in infinite dimensions any trace-norm hood of a state ρ of finite entropy contains a dense set of states of infinite entropy. These cannot be near ρ in any physical sense. We would also expect that a viable geometric description of the dynamics of a macroscopic system would be a path $\{\rho(t), t \geq 0\}$, where the relative entropy

$$S(\rho(t)|\rho(0)) := \text{Tr } \rho(t)(\log \rho(t) - \log \rho(0)) \quad (1)$$

is finite. This is so because the relative entropy is closely related to the free energy. These requirements cannot be controlled by the trace norm. We propose a stronger topology for the manifold, controlled by a new norm which we call the Araki norm, as it arises naturally in H. Araki's theory of 'expansionals'. This is sufficient for the relative entropy between a state and any of its neighbours to be finite, as well as ensuring that the mixture connection, and the canonical connection, are defined on the manifold. They are then dual relative to the Bogoliubov-Kubo-Mori metric. Our condition is far from being necessary for these results, however; so we can expect that this is not the last word on dual information geometry.

We start with §1.2, a review of our definition [10, 11] of the quantum information manifold in infinite dimensions. This makes use of the theory of relatively bounded perturbations, and constructs a manifold with a metric and the $(+1)$ -affine structure. In § 2 we note that the (-1) -affine structure can be defined for a vector subspace of the tangent space defined

there, namely, for nearby states. We say that a density matrix σ is nearby the density matrix ρ if there exists a constant $C > 1$ such that

$$C^{-1}\rho \leq \sigma \leq C\rho \quad (2)$$

in the sense of operator inequality.

In the context of faithful states on von Neumann algebras, Araki had proved [2] that (2) is enough to ensure that the relative Hamiltonian of ρ and σ is a bounded element of the algebra, with a holomorphic property. Our theory is in a more concrete setting, in which these results are true but not so deep. By establishing the converse of Araki's theorem in our simple case, we are able to show that the (+1)-affine structure is retained on this subspace, which therefore has two flat affine structures.

1.2 The quantum information manifold

Let \mathcal{H} be a separable Hilbert space, and $B(\mathcal{H})$ the W^* -algebra of bounded operators on \mathcal{H} , with norm $\|\cdot\|$. Recall [10] that the quantum information manifold for \mathcal{H} is constructed as follows. Let \mathcal{C}_p , $0 < p < \infty$ be the vector space of $A \in B(\mathcal{H})$ such that

$$\|A\|_p := \text{Tr}(|A|^p)^{1/p} < \infty. \quad (3)$$

Then \mathcal{C}_p is the Schatten class, a Banach space, for $1 \leq p \leq \infty$ and is a complete quasinormed space for $0 < p < 1$ [9]. Let Σ be the convex set of (von Neumann) density operators on \mathcal{H} . Then we defined the underlying set of the information manifold to be

$$\mathcal{M} = \bigcup_{0 < p < 1} \mathcal{C}_p \cap \Sigma. \quad (4)$$

Thus \mathcal{M} is the set of density operators ρ such that there exists $p \in (0, 1)$ such that ρ^p is of trace-class. Given $\rho_0 \in \mathcal{M}$ we suggested [10, 11, 6] various norm topologies on \mathcal{M} for points near ρ_0 as follows: write $\rho_0 = \exp\{-H_0 - \Psi_0\}$, where $H_0 \geq 0$, and let X be a quadratic form defined on $Q(H_0) := \text{Dom}(H_0^{1/2})$, such that for some $\epsilon \in [0, 1/2]$ we have

$$\|X\|_\epsilon := \|(H_0 + I)^{-1+\epsilon} X (H_0 + I)^{-\epsilon}\| < \infty. \quad (5)$$

We showed that if this norm is small enough, $H_0 + X$ defines a self-adjoint operator, that

$$\rho_X := \exp\{-H_0 - X - \Psi_X\} \in \mathcal{M} \quad (6)$$

holds, that the generalised expectation $\rho_0 \cdot X$ can be defined, and adjusted to zero, which determines the normalising constant Ψ_X . Then the partition function $Z_X := e^{\Psi_X}$ was shown to be a Lipschitz continuous function of X in the norm $\|\cdot\|_\epsilon$, if $\epsilon = 1/2$, and C^∞ in the Fréchet sense, and analytic in the canonical coordinates, if $0 \leq \epsilon < 1/2$. We showed that topologised in this way around each $\rho_0 \in \mathcal{M}$, \mathcal{M} becomes a Banach manifold. Each 'hood of a point is a convex set, under the convex structure $X_1, X_2 \mapsto \lambda X_1 + (1 - \lambda)X_2$, $\lambda \in (0, 1)$. This defines the (+1)-affine structure on \mathcal{M} .

So far, we have been unable to prove that (for some choice of ϵ) if $\rho, \sigma \in \mathcal{M}$ then $\lambda\rho + (1 - \lambda)\sigma \in \mathcal{M}$. Thus, the (-1) -affine structure is missing. In the next section we find a subspace on which both the $(+1)$ -affine structure and the (-1) -affine structures are defined; this turns out to be the same space as used by Araki in the much more general case of *KMS*-states. It is not complete in any of the epsilon norms; however, it is complete in a norm arising in Araki's theory.

2 Nearby states and the (-1) -affine structure

It is convenient to extend the states by scaling, to get finite weights on $B(\mathcal{H})$. We denote the extended space of states by $\tilde{\Sigma}$ and the extended manifold by $\tilde{\mathcal{M}}$. We write a finite weight as $\rho_X = \exp(-H_0 - X)$.

We shall also use a more general concept than being nearby:

Definition 1 *Suppose that there exists $C > 1$ and $p \in [0, 1)$ such that in the sense of operator inequalities,*

$$C^{-1}\rho^{1+p} \leq \sigma \leq C\rho^{1-p}. \quad (7)$$

Then we say that σ is p -nearby ρ .

It is clear that in the case $p = 0$ this is an equivalence relation. This case reduces to (2), so that σ and ρ are *nearby* each other.

Let $\rho_0 \in \tilde{\Sigma}$ and suppose that ρ_X is nearby ρ_0 . Then Araki [2] proved that X is bounded. In our special setting, this is a corollary to an easy result.

Theorem 2 *If $\rho_0, \rho_X \in \tilde{\Sigma}$ and ρ_X is p -nearby ρ_0 , then X is an H_0 -bounded form, with form bound $\leq p$.*

PROOF. The map $X \mapsto \log X$ is an operator monotone map, which also applies to forms. Taking logs of (7) we get as an inequality of forms

$$-\log C - (1 + p)H_0 \leq -H_0 - X \leq \log C - (1 - p)H_0.$$

This shows that $Q(X) \subseteq Q(H_0)$, so the forms X and H_0 are comparable, and we may cancel H_0 to get

$$-\log C - pH_0 \leq -X \leq \log C + pH_0.$$

Thus, X is an H_0 -bounded form, with form bound $\leq p$; if $p = 0$ we see that X is bounded by $\log C$. QED

Suppose now that σ_1 and σ_2 are finite weights, and both are p -nearby the finite weight ρ . Then $\lambda\sigma_1 + (1 - \lambda)\sigma_2$ is p -nearby ρ . This follows easily from the definition (7). Since any state is p -nearby itself, we see that the set of states that are p -nearby ρ_0 form a mixture family. The problem is to find a norm topology (relative to the $(+1)$ -linear structure) such that all (-1) -mixtures lie in the 'hood of the state ρ in that norm topology.

3 Araki's analyticity and its converse

Araki [2] showed in the context of *KMS* theory, that if ρ_X is nearby the state ρ , then the map

$$t \mapsto \rho^t X \rho^{-t}$$

is bounded and holomorphic in the circle $\{t \in \mathbf{C} : |t| < 1/2\}$. (8)

This suggests the definition of the Araki norm:

Definition 3 For any $\rho \in \mathcal{M}$ and any bounded operator X ,

$$\|X\|_A := \sup_{0 \leq |t| < 1/2} \left\{ \|\rho^t X \rho^{-t}\| \right\}.$$

In our context, we prove a converse, to get

Theorem 4 Suppose that $X \in B(\mathcal{H})$, and that ρ and ρ_X are finite weights. Then (8) holds if and only if ρ_X is nearby ρ .

PROOF. From Araki's result, we only need to prove the 'only if' direction. We start with the Matsubara expansion

$$\rho_X = \rho + \sum_{n=1}^{\infty} \frac{1}{n!} \int_0^1 \prod_{j=1}^{n+1} d\alpha_j \delta \left(\sum_{i=1}^{n+1} \alpha_i - 1 \right) \rho^{\alpha_1} X \rho^{\alpha_2} X \dots X \rho^{\alpha_{n+1}} \quad (9)$$

and obtain a formal expansion for the quadratic form

$$\begin{aligned} \rho^{-1/2} \rho_X \rho^{-1/2} &= 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_0^1 \prod_{j=1}^{n+1} d\alpha_j \delta \left(\sum_{i=1}^{n+1} \alpha_i - 1 \right) \left(\rho^{\alpha_1 - 1/2} X \rho^{1/2 - \alpha_1} \right) \\ &\quad \left(\rho^{\alpha_1 + \alpha_2 - 1/2} X \rho^{1/2 - \alpha_1 - \alpha_2} \right) \dots \left(\rho^{1/2 - \alpha_{n+1}} X \rho^{\alpha_{n+1} - 1/2} \right). \end{aligned} \quad (10)$$

Each of the components $\pm(\alpha_1 + \dots + \alpha_j - 1/2)$, $1 \leq j \leq n$ lies between $-1/2$ and $1/2$, so by (8) each term is bounded by $M^n/n!$, where $M = \|X\|_A$, defined in (8). So the series is norm convergent to a bounded operator of norm $\leq C := e^M$. The series is then known to converge to $\rho^{-1/2} \rho_X \rho^{-1/2} \leq C$, giving $\rho_X \leq C\rho$.

In the other direction we consider the unbounded positive operators ρ_X^{-1} and ρ^{-1} related by the formal series

$$\rho_X^{-1} = \rho^{-1} + \sum_{n=1}^{\infty} \frac{1}{n!} \int_0^1 \prod_{j=1}^{n+1} d\alpha_j \delta \left(\sum_{i=1}^{n+1} \alpha_i - 1 \right) \rho^{-\alpha_1} X \rho^{-\alpha_2} \dots X \rho^{-\alpha_{n+1}}. \quad (11)$$

Multiplying on the left and right by $\rho^{1/2}$ it becomes a series of bounded operators:

$$\begin{aligned} \rho^{1/2} \rho_X^{-1} \rho^{1/2} &= 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_0^1 \prod_{j=1}^{n+1} d\alpha_j \delta \left(\sum_{i=1}^{n+1} \alpha_i - 1 \right) \left(\rho^{1/2 - \alpha_1} X \rho^{\alpha_1 - 1/2} \right) \\ &\quad \left(\rho^{1/2 - \alpha_1 - \alpha_2} X \rho^{\alpha_1 + \alpha_2 - 1/2} \right) \dots \left(\rho^{\alpha_{n+1} - 1/2} X \rho^{1/2 - \alpha_{n+1}} \right). \end{aligned} \quad (12)$$

Again, each term is bounded by $M^n/n!$ so we get, as forms

$$\rho^{1/2} \rho_x^{-1} \rho^{1/2} \leq e^M I, \text{ so } \rho_x^{-1} \leq C \rho^{-1}.$$

Now, $A \mapsto A^{-1}$ is an operator-monotone decreasing map, so we get

$$\rho_x \geq e^{-M} \rho = C^{-1} \rho. \quad \text{QED.}$$

From this we see if the the hood of a state ρ be taken to be the subset of \mathcal{M} consisting of states ρ_x that are nearby ρ_0 , then the tangent space at ρ carries both the canonical and the mixture affine structures. This follows from this theorem, and the obvious fact that (+1)–mixtures of states having the property (8) also have that property.

One can immediately see that the norm given by the *BKM*-metric is weaker than the Araki norm:

$$\|X\|_M^2 = \int_0^1 \text{Tr} \{ \rho^t X \rho^{1-t} X \} dt = \int_0^1 \text{Tr} \{ \rho^{t-1/2} X \rho^{1/2-t} \} dt \leq \|X\|_A^2.$$

It can be shown that it is properly weaker; in our proposed topology, the manifold is not complete in the *BKM* metric, and so also not complete in any epsilon norm, or the Orlicz norm suggested in [12].

4 A Banach manifold with canonical-mixture duality

Let us start with the manifold \mathcal{M} , of [10] as a set, but define its topology using the Araki norm rather than one of the epsilon norms. We extend the manifold by allowing finite faithful weights as well as faithful states. If ρ is a finite faithful weight, it can be written as $\rho = \exp\{-H_0\}$. We allow as points of the hood of ρ all the nearby weights. These are all finite and faithful. The points in a hood form a (−1)–mixture family. By Theorem(4), the family is parametrized by those bounded operators X such that $\rho^t X \rho^{-t}$ is bounded and holomorphic in $|t| < \frac{1}{2}$. This family is convex under the canonical, that is, (+1)–affine structure, since if X and Y obey this condition, so does $\lambda X + (1 - \lambda)Y$. More, the space is complete in the Araki norm (using the canonical affine structure to define the uniformity). For, a sequence of functions holomorphic and bounded in an open disc, which is a Cauchy sequence in the sup norm, converges to a bounded holomorphic function in the same disc. It has two flat affine structures, the canonical (+1) and the mixture, (−1). We now show that they are dual relative to the *BKM* metric.

Theorem 5 (Duality) *The (±)–affine structures defined here are dual relative to the BKM metric.*

PROOF. By construction, the relative entropy $S(\rho_0|\rho_x)$ between two nearby states ρ_0 and ρ_x is finite, and by the result of [11], it is smooth. By definition, the *BKM* metric is half of the second (Fréchet) derivative of this. This is also the metric obtained as half of the Hessian of the Legendre dual, the other relative entropy $S(\rho_x|\rho_0)$. Then g is the Hessian of

$$(S(\rho_0|\rho_x) + S(\rho_x|\rho_0)) = -\text{Tr} \{ X(\rho_0 - \rho_x) \},$$

by a simple calculation. Now X is the displacement in the state along a $(+1)$ -geodesic, and $\rho_0 - \rho_X$ is minus the displacement along a (-1) -geodesic. If $\rho_0 - \rho_X := \delta\rho$ is small in the Araki norm, the left-hand side is close to $g(\delta\rho, \delta\rho)$ and the right-hand side is $\text{Tr}\{\delta\rho_+ \delta\rho_-\}$. This is the mixed representation of the BKM -metric, and the equation expresses the duality.

QED

We may furnish the tangent space of \mathcal{M} with the Araki norm, for bounded holomorphic operators $X(t) = \rho^t X \rho^{-t}$ in the disc $|t| < \frac{1}{2}$. This defines a hood of ρ . The norms in overlapping regions, around ρ_0 and ρ_1 , are compatible; this requires the equivalence of the norms

$$\sup_{|t| < 1/2} \{\|\rho_0^t X \rho_0^{-t}\|\} \quad \text{and} \quad \sup_{|t| < 1/2} \{\|\rho_1^t X \rho_1^{-t}\|\}.$$

This is true, since they are related by conjugation with the Connes cocycle $\rho_0^t \rho_1^{-t}$, and this is bounded and analytic in the disc [8]. In this way, we have a (smallish) Banach manifold with the dual structure.

We can see that the trace norm on states is weaker than the norm $\|X\|$ on the space of perturbations, and therefore also weaker than the Araki norm. Indeed,

Theorem 6 *If $\rho = \rho_0$ and $\sigma = \rho_X$, then*

$$\|\rho - \sigma\|_1 \leq \|X\|.$$

PROOF. Our equality

$$S(\rho|\sigma) + S(\sigma|\rho) = \text{Tr}\{(\rho - \sigma)X\}$$

shows that (by the quantum Kullback inequality [7],

$$\|\rho - \sigma\|_1^2 \leq \text{Tr}\{(\rho - \sigma)X\} \leq \|\rho - \sigma\|_1 \|X\|$$

So dividing by $\|\rho - \sigma\|_1$ gives the theorem. **QED**

The Araki norm

$$\|X\| \leq \|X\|_A := \sup_{|t| < 1/2} \{\|\rho^t X \rho^{-t}\|\}$$

is thus also stronger than the trace norm. The expansional for the relative entropy shows that all states σ in an Araki 'hood of a point $\rho \in \mathcal{M}$ has finite relative entropy $S(\rho|\sigma)$, which was the desired property.

Acknowledgements: This work was done during a research visit to the Science University of Tokyo, Noda. I thank M. Ohya for the kind invitation, and N. Watanabe and H. Hasegawa for useful discussions.

References

- [1] Amari, S.-I., and Nagaoka, H., *Methods of Information Geometry*, Amer. Math. Soc. Providence RI, translated from the 1993 Japanese by D. Harada.
- [2] Araki, H., Publ. Res. Inst. Math. Sci., Kyoto University, **9**, No.1, 1973.

- [3] Gibilisco, P., and Isola, T., Connections on statistical manifolds of density operators by geometry of non-commutative L^p -spaces, *Infin. Dimens. Anal. Quantum Prob. Relat. Topics*, **2**, 169-178, 1999.
- [4] Gibilisco, P. and Pistone, G., Connections on non-parametric statistical manifolds by Orlicz geometry, *Infin. Dimens. Anal. Quantum Probab. Related Topics*, **1**, 325-347, 1998.
- [5] Grasselli, M. R. *Classical and Quantum Information Geometry*, Ph. D. thesis, King's College London, 2001.
- [6] Grasselli, M. R., and Streater, R. F., The quantum information manifold for epsilon-bounded forms, *Rep. Math. Phys.*, **46**, 325-335, 2000. arXiv/math-ph/9910031.
- [7] Hiai, F., Ohya, M. and Tsukuda, M., Sufficiency, *KMS* condition and relative entropy in von Neumann algebras, *Pacific J. Math.*, **96**, 99-109, 1981.
- [8] Ohya, M., and Petz, D., **Quantum Entropy and its Use**, Springer-Verlag, 1993.
- [9] Pietsch, A., **Nuclear Locally Convex Spaces**, Springer-Verlag, 1972.
- [10] Streater, R. F., The information manifold for relatively bounded potentials, *Trudy Mat. Inst. Steklova*, **228**, 217-235, 2000, in English. Also in the English ed. *Proc Steklov Inst. Maths.*, **228**, 205-223, 2000. arXiv/math-ph/9910035.
- [11] Streater, R. F., *The analytic quantum information manifold*, in **Stochastic Processes, physics and geometry: new interplays** Eds. F. Gesztesy et al., Canadian Math. Soc., Conference Proc **29**, 603-611, 2000. arXiv/math-ph/9910036.
- [12] Streater, R. F., Quantum Orlicz Spaces, to appear.